17.5. HARMONIC SYNTHESIS AND SUPERPOSITION*

Let $\mathcal{F}\ell^1$ be the algebra of all functions f, defined on the circumference $\mathbb{T}$ and which can be expanded in absolutely convergent Fourier series:

$$f(\zeta)=\sum_{n\in\mathbb{Z}} \hat{f}(n)\zeta^n,\ \zeta\in\mathbb{T};\quad \|f\|_1 \stackrel{def}{=} \sum_{n\in\mathbb{Z}} |\hat{f}(n)|<+\infty.$$

Let $f\in\mathcal{F}\ell^1$ and assume, for the sake of definiteness, that $f(\mathbb{T})=[-1,1]$. We say that f *admits harmonic synthesis* [1] if there exists a sequence of functions $(\varphi_n)_{n\geq 1}$, $\varphi_n\in\mathcal{F}\ell^1$, such that $\lim_n \|\varphi_n - f\|_1 = 0$ and $f^{-1}(0)\subset \operatorname{Int}\varphi_n^{-1}(0)$, n = 1, 2,... .

All sufficiently smooth functions admit synthesis but, by P. Malliavin's theorem, in $\mathcal{F}\ell^1$ there exist functions which do not admit synthesis.

Question 1. Assume that f admits synthesis; can one select a sequence $(\varphi_n)_{n\geq 1}$ such that $\varphi_n = F_n\circ f$, where F_n are functions defined on [−1, 1]?

We consider the set [f] of all functions F, defined on [−1, 1], for which $F\circ f\in\mathcal{F}\ell^1$. This is a Banach algebra with the norm $\|F\|_{[f]} = \|F\circ f\|_1$. It contains the identity mapping x of the segment [−1, 1]. Now, our question can be reformulated in the following way.

Question 2. Assume that f admits synthesis; does it follow from here that x can be approximated in the normed algebra [f] by functions which vanish in the neighborhood of the point 0?

If it turns out that $[f]\subset C^1[-1,1]$ (the imbedding is necessarily continuous by Banach's theorem), then the functional $\delta': F\to F'(0)$ will separate x from all these functions. Therefore, a special case of our problem is the following question.

Question 3. Assume that f admits synthesis; can we have

$$[f]\subset C^1[-1,1]?$$

Since $\|e^{2\pi i n x}\|_{C^1[-1,1]}\asymp n$ for $n\to+\infty$, a further specialization leads to Question 4.

Question 4. Can one construct a function f admitting synthesis and for which $\lim_n \frac{1}{n}\times \|e^{inf}\|_1 > 0$?

Presently (1978), very little is known about the structure of the ring [f]. Theorems of the Wiener–Lévy type ([1], Chap. VI; [2]) give sufficient conditions for the inclusion $F\in[f]$, which, however, are much stronger than the C^1-smoothness of the function F. On the other hand, assume that the function $t\to f(e^{it})$ is even on [−π, π] and one-to-one on [0, π]. Then, any even function on [−π, π] has the form $F\circ f$ and the answer to our Question 1 is in the affirmative. Therefore $[f]\not\subset C^1$ and the known Wiener–Lévy type theorems for f are trivially coarse. Finally, Kahane [3] has constructed examples of such functions f that $[f]\subset C^\infty[-1,1]$. In other words, the ring [f] has to be considered a completely unstudied object.

One can attempt to obtain the answer to Question 3 in the following manner. If $[f]\subset C^1$, then the functional $\delta': F\to F'(0)$ is defined on [f] and generates a functional $\delta'(f)$ on the subalgebra $[[f]]=\{F\circ f : F\in[f]\}$. If one could succeed to extend this functional to all of $\mathcal{F}\ell^1$ so that $\langle\delta'(f), g\rangle = 0$ for $f^{-1}(0)\subset \operatorname{Int} g^{-1}(0)$, then the absence of synthesis for f would be proved. Some hope is suggested by the circumstance that Malliavin's lemma [1] regarding the absence of synthesis is proved precisely in this manner. Namely, under the

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additional condition $\int_{\mathbb{R}_+} u \| e^{iuf} \|_{(\mathcal{F}\ell^1)^*} du < +\infty$, $\int_{\mathbb{T}} dm \int_{\mathbb{R}} e^{iuf} du = 1$, the Malliavin functional

$$g \longrightarrow \int_{\mathbb{T}} dm \int_{\mathbb{R}} iue^{iuf} g\, du, \quad g \in \mathcal{F}\ell^1,$$

furnishes the continuation of the functional δ' from the subalgebra [[f]], possessing the required properties.

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LITERATURE CITED

1. J.-P. Kahane, Séries de Fourier Absolument Convergentes, Springer-Verlag, New York (1970).
2. E. M. Dyn'kin, "Wiener–Lévy type theorems and estimates for Wiener–Hopf operators," Mat. Issled., 8, No. 3, 14-25 (1973).
3. J.-P. Kahane, "Une nouvelle réciproque du théorème de Wiener–Lévy," C. R. Acad. Sci. Paris, 264, A104-106 (1967).